

# CONCEPTS FOR A THEORY OF THE ELECTROMAGNETIC FIELD\*

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## Abstract

The object of this contribution is twofold. On one hand, it rises some general questions concerning the definition of the electromagnetic field and its intrinsic properties, and it proposes concepts and ways to answer them. On the other hand, and as an illustration of this analysis, a set of quadratic equations for the electromagnetic field is presented, richer in pure radiation solutions than the usual Maxwell equations, and showing a striking property relating geometrical optics to all the other Maxwell solutions.

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## 1 Introduction

I think that the principal mission of the scientific culture is not to increase our *knowledge*, as it is frequently stated, but to ameliorate our *understanding* of the world. It is clear that, although frequently connected, and sometimes intimately, these two goals present deep differences in content and in extent.

Scientific culture, like any other human culture, when absorbed without reflection, is also alienation. In particular, the present state of the classical electromagnetic theory shows

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abundantly this feature. And it is trying to escape a little to this alienation on the subject that my collaborators and myself built up some concepts and results, of which a part is presented here [1].

This contribution wishes to ameliorate our understanding of the electromagnetic field, and consists of very simple concepts, arguments and propositions. But, even simple, these elements present a certain interest: they rise pertinent questions on classical fields, outline answers, propose new formalisms and lead directly to new equations with striking properties. It is about these points that I would like to talk here.

Here we are interested by the *texture* of the *physical* electromagnetic fields [2].

The first problem that a field theory faces, be it electromagnetic or not, is the adequacy between the physical phenomena trying to be described by the theory and the solutions that the theory provides. Some aspects of this problem, and in particular the elements that make today not one-to-one the relation between physical phenomena and field theories, are commented in Section 2.

The second problem is that of the adequacy between the physical quantity describing the field itself (here the electromagnetic field) and the mathematical object chosen by the theory to represent this quantity (vectors for the electric and magnetic fields, anti-symmetric tensor for the total electromagnetic one). Section 3 analyses briefly this problem and points out its experimental character.

By texture of an electromagnetic field we mean the particular link among the ingredients of the mathematical object chosen to represent this electromagnetic field, as well as among its gradients, specially the invariant or intrinsic ingredients. After remembering the notions of *regular* and *pure radiation* electromagnetic fields, those of *proper* energy density of an electromagnetic field and of observer *at rest* with respect to a regular electromagnetic field are presented in Section 4. It also presents the intrinsic relation between the electromagnetic field and its *principal* directions, the *physical* interpretation of them in terms of the electric and magnetic fields with respect to any observer, and the one-to-one *decomposition* of the electromagnetic energy tensor in two pure radiation components. It follows a method allowing, for the first time, to obtain the necessary and sufficient conditions to be imposed on an electromagnetic field in order to admit eigen-directions with prescribed differential properties. Some comments about the general form of the *charge scaling* law and the *superposition* law for any electromagnetic theory are added.

It is usually ignored that, whatever be the tensorial object describing an ordinary electromagnetic field, a strong one *could* change not only its *intensity* but even its *specific* tensor character. In Section 5 a formal example is given in which strong electromagnetic fields would be given by Lorentz tensors, and some results necessary to handle this eventuality are presented.

Finally, in Section 6, as a precise application of the above concepts, a quadratic generalization of Maxwell equations is presented. These new equations for the electromagnetic field admit “more light” (pure radiation solutions) than the Maxwell ones, but *exactly* the same charged electromagnetic fields (regular solutions). They have the striking property, on one hand, of being a geometric optics approximation of Maxwell equations and, on the other hand, of containing the exact Maxwell equations themselves.

## 2 Phenomena and Field Theories

a) For Newton (*resp.* Maxwell) classical field theory, the field outside the sources in a finite region of the space (*resp.* space-time) determines completely the masses (*resp.* charges) that produce it, and consequently the field everywhere (*resp.* in all the causal past of the region). For this reason, we call here for short *pure field theories* the local versions of these theories, and of their possible local generalizations (including General Relativity), at the exterior of the sources; that is to say, the set of regular solutions in a local domain of the corresponding differential equations in vacuum [3] [4].

Denote by  $\mathbf{F}$  the set of all physical phenomena of a certain class and by  $\mathbf{S}$  the set of (physical or unphysical) phenomena described by solutions to a field theory for this class. Obviously, the set of physical phenomena described by such a field theory,  $\mathbf{F} \cap \mathbf{S}$ , generically differs from the whole set  $\mathbf{F} \cup \mathbf{S} : \mathbf{F} \cap \mathbf{S} \subset \mathbf{F} \cup \mathbf{S}$ . Thus, generically  $\mathbf{F}$  contains a subset  $\hat{\mathbf{F}}$  of physical phenomena not described by the field theory,  $\hat{\mathbf{F}} \equiv \mathbf{F} - \mathbf{F} \cap \mathbf{S}$ , as well as  $\mathbf{S}$  contains a subset  $\hat{\mathbf{S}}$  of solutions to the field theory which are unphysical,  $\hat{\mathbf{S}} \equiv \mathbf{S} - \mathbf{F} \cap \mathbf{S}$ , so that we have  $\mathbf{F} \cup \mathbf{S} \equiv \hat{\mathbf{F}} \cup (\mathbf{F} \cap \mathbf{S}) \cup \hat{\mathbf{S}}$ .

Theories for which the subset  $\hat{\mathbf{F}}$  vanishes are called *complete*. Theories for which the subset  $\hat{\mathbf{S}}$  vanishes are called *strict*.

b) For non-complete theories, the analysis of the subset  $\hat{\mathbf{F}}$  of phenomena not described by field theories sets out, in general, a problem at once of observation, experimentation and inferential logic. But, in particular, some cases may be analysed (at least partially) with the standard methods of theoretical physics. Among these cases, one may mention the possible existence of space-time metrics insensitive to the polarization of gravitational waves. If one accepts that the curvature of the space-time due to the presence of a gravitational field follows a “universal law of gravitational deformation”, then one can locally separate unambiguously the metric from the gravitational field, which turns out to be given by a two-form [5]. One can then show that the polarization of a gravitational wave is not “detected” by the space-time metric.

Another case able to be analysed theoretically is that of charge-independent and non-radiation electromagnetic fields. One knows that, under Maxwell equations, to shake a charged particle allows to detach parts of its field, which get away at the velocity of light (radiation field component of charged accelerated particles). The hypothesis that strong electromagnetic fields are better represented by Lorentz tensors than by two-forms, offers the possibility to detach from particles non-escaping (non-radiation) parts of the field.

Also, there exist arguments to suspect that the solutions to Maxwell equations representing light could be insufficient. This possible lack of sufficiency of Maxwellian light may be defined precisely, and equations avoiding it may be obtained.

At present, of these three examples, the second one is the more speculative [6], and although in Section 5 we give some elements to handle Lorentz tensors related to two-forms, it will not be study in depth. The first one has been briefly presented in [5], and needs a critical and comparative analysis with the standard points of view. Finally, the third example has been analysed in [7], and will be explained in some detail in the last section of

this paper.

c) Contrarily to an extended opinion, the unphysical solutions  $\hat{\mathbf{S}}$  of non-strict field theories fill in, roughly spoken, almost all the space of solutions of the field equations, the solutions  $\mathbf{F} \cap \mathbf{S}$ , that describe physical phenomena being a set of “very null” measure [8].

This situation is due to the existence of several mechanisms that generate unphysical solutions. Among them, the more important ones are:

- *Negative masses*: in gravitational theories, the solutions to field equations depend on a set of constants that are related to the masses of the system, but the (Newton or Einstein) gravitational field theories do not contain neither algorithms nor constraint insuring that all the corresponding masses of their exterior solutions are positive. For example, in Newton gravitational theory, the number of unphysical solutions with a finite number  $N$  of singularities (corresponding to a distribution of  $N$  masses such that at least one of them is negative) associated to every physical solution is given by  $\sum_{i=1}^N \binom{N}{i} = 2^N - 1$  [9].

- *Duality invariance*: Maxwell equations at the exterior of sources are invariant by duality rotations, i.e. by transformations of the form  $F' = \cos \phi F + \sin \phi *F$ . They are such that the sum of arbitrarily duality rotated physical fields is not in general a duality rotation of a physical field. Duality rotations introduce magnetic monopole charges but we do not know neither algorithms nor constraint equations allowing to know if a given (local, exterior) solution  $F$  to the field equations is able or not to generate, by a duality rotation, a physical field. Now, for fields admitting a finite number of singularities, the number of unphysical solutions that correspond to every physical one is  $R^N$  [9].

- *Advanced-retarded symmetry*: the principle of causality and the finite character of the velocity of perturbations in relativistic field theories lead to consider physical phenomena as generated by retarded interactions (electromagnetism and relativistic gravitation). But Maxwell equations (and in some sense Einstein ones) admit indistinctly retarded and advanced solutions, and no general algorithms or constraint equations are known to distinguish them. Furthermore, Maxwell theory being linear, arbitrarily weighted sum of advanced and retarded solutions,  $\lambda F_{\text{retarded}} + \mu F_{\text{advanced}}$ , is a new solution, and one has also  $R^N$  unphysical solutions corresponding to every physical one admitting  $N$  singularities [9].

In fact, we may conclude that *there are not known complete and strict classical pure field theories*. The only known example of a strict theory is the little Newtonian gravitational theory of one point particle [10]; in spite of its restricted character, this example is heuristically very rich, and allows to have an inkling of what a complete and strict pure field theory looks like and in what situations it may be useful. It is very striking that, apart from this particular example, no other tentative had taken place since the creation of the concept of field.

d) The construction of a new field theory, with or without the above characteristics and whatever be the motivations (to avoid singularities of their solutions, to include in field form the equations of motion, to take directly into account terms of self-interaction, to include additional pure radiation solutions, etc) needs the following two tasks:

- \* to represent the physical field quantities by pertinent mathematical objects,
- \* to find convenient differential equations for them.

Paradoxically, the first of these points has been systematically ignored. For this reason, the following section presents comments on some aspects related to it.

### 3 Mathematical Representation of Physical Fields

**a)** In classical physics, the points of our physical space or space-time are (locally) identified with the points of the mathematical spaces  $R^3$  or  $R^4$ . For this reason, the vector character of, say, the position vector is only a matter of mathematical definition.

But, once this identification is made, the *particular* tensor character, at every one of these points, of *any* physical quantity has to be theoretically *founded* and experimentally *verified* [11].

Thus, the adequacy of a physical quantity with its formal or mathematical representation involves:

- the consideration of the offer of mathematical objects: scalars, vectors, tensors, spinors, etc,
- the good comprehension of their invariant ingredients as well as of the structure involving these invariants and
- the experimental confrontation necessary to guarantee that the correct choice of representation has been made.

**b)** Concerning the first of these points, it is important not to forget, as it is frequent, that the notion of 'tensor field' is always attached to a group, although frequently implicit. Thus, in Newtonian mechanics the *acceleration* of a particle is a vector for the whole Galileo group  $G$  of coordinate transformations between inertial observers, its *velocity* is a vector for the restricted group  $I$ ,  $I \subset G$ , of coordinate transformations leaving invariant (internal or adapted) a given inertial observer, and its *position* is only a vector for the more restricted group  $I_0$ ,  $I_0 \subset I \subset G$ , of coordinate transformations that leave the origin  $O$  unchanged. Although trivial, this example shows that, when exploring new, enlarged situations, the invariance group of (the mathematical representation of) a physical quantity has to be analysed carefully. In Special Relativity, this has not been the case, for example, in generalizing Maxwell equations from inertial observers to accelerated ones.

The electric field  $e$  and the magnetic field  $h$  measured by an inertial observer are assumed [12] to be vectors under the above mentioned group  $I$  of coordinate transformations adapted to that observer. But they are not vectors under the whole Poincaré group  $P$ , of coordinate transformations between arbitrary inertial observers. As it is well known, there does not exist any function of the sole electromagnetic quantities  $e$  and  $h$  that be a tensor under  $P$ . In order to construct a tensor under  $P$ , two important features are needed. The first one is the *addition*, to the two quantities  $e$  and  $h$ , of the (unique) kinematical quantity characterizing the inertial observer: its unit velocity  $u$ . The second one is the *substitution*

of the search of the vector character of the two initial ingredients  $e$  and  $h$  by that of the anti-symmetric tensor character of a sole function  $F$  of the three ingredients  $e$ ,  $h$  and  $u$ . Both features lead to the well known result  $F \equiv F(e, h, u) = u \wedge e - *(u \wedge h)$  [14].

Faced to such a denouement and in such a spirit, the extension of the above electromagnetic quantity  $F$  to a larger group of accelerated observers (be it in Minkowski space-time or in the curved ones of General Relativity) involves the following two physical questions: is the electromagnetic field  $F_a$  measured by an accelerated observer a function  $F_a(e, h, u)$  of its kinematical quantity  $u$  alone, or does it depend also on its (now non vanishing) acceleration  $a$ ,  $F_a = F_a(e, h, u, a)$  ? does the electromagnetic field quantity  $F_a$  measured by an accelerated observer remain a second order anti-symmetric tensor? [15].

Neither of the principles of relativity, covariance or minimal coupling, in their usual formulations, allow to give a clear answer to these questions. In fact, we have no other arguments that mathematical simplicity or physical dogmatism to clearly eliminate contributions of the above two aspects on the mathematical representation of the electromagnetic field quantities on the space-time.

c) But even the assumption that the electric and magnetic fields measured by an inertial observer are vectors under his adapted group  $I$  has to be submitted to experimental agreement. The assertion that they are vectors means that, if we measure the force  $\mathbf{f}_\alpha$  needed to cancel them at different angles  $\alpha$ , we must obtain the cosines law  $\mathbf{f}_\alpha = \mathbf{f}_0 \cos \alpha$ .

Up to what precision such a law is true for the electric and/or the magnetic fields?

Observe that a law different from the cosines law, even very slightly different, will oblige to represent these fields by means of geometric objects drastically different from vectors (although analytically related [13]).

Experiments such as, for example, the measure of the ratio inertial/gravitational mass of a body are undoubtedly interesting; but those trying to measure the adequacy of the laws associated to the vector or tensor character of the fundamental electromagnetic fields (among others) would provoke a similar interest. Unfortunately, this is not the case at present.

## 4 Intrinsic Elements of the Electromagnetic Field

a) In spite of the above comments, we shall suppose here, unless otherwise stated, that an electromagnetic field in the space-time is (locally) described by a two-form  $F$  (second order anti-symmetric covariant tensor field) such that, if  $u$  is the unit velocity of an arbitrary observer, the electric and magnetic fields for him are given by

$$e = i(u)F \quad , \quad h = i(u) * F \quad , \quad (1)$$

where  $i$  stands for the interior product and  $*$  for the Hodge operator associated to the space-time metric  $g$ . Then, one has equivalently  $F = u \wedge e - *(u \wedge h)$  [14], where  $\wedge$  stands for the exterior product.

For many technical uses, it is sufficient to work with this two-form  $F$ . But, at every point of the space-time,  $F$  is an element of the tensor algebra over the real four dimensional

vector space. Consequently,  $F$  cannot be but a subset of vectors and numbers at every point, that is to say, a set of vector fields and scalar functions on the space-time. In the tensor formalisms, these ingredients are called the *invariants* or *intrinsic elements* of  $F$  [16].

b) It is well known that the independent scalar functions of  $F$  are two, usually chosen as

$$\phi \equiv \text{tr } F^2 \quad , \quad \psi \equiv \text{tr } F * F \quad , \quad (2)$$

where  $\text{tr}$  is the trace operator, and one has the relations  $\phi = 2(h^2 - e^2)$ ,  $\psi = -4(eh)$ , which reveal their implication on the fields  $e$  and  $h$ ; they allow fixing, in the plane determined by these fields, one of them as a function of the other.

But what it seems not known is the implications of these scalars on the energy variables.

Remember that the physical components with respect to an observer  $u$  of the Minkowski energy tensor  $T$  of  $F$ ,

$$T = \frac{1}{2}(F^2 + (*F)^2) \quad (3)$$

are the *energy density*  $\rho$ , the *Poynting vector*  $s$  and the *stress tensor*  $\tau$ , respectively given by

$$\rho \equiv i^2(u)T \quad , \quad s \equiv \perp(u)i(u)T \quad , \quad \tau \equiv \perp(u)T \quad , \quad (4)$$

where  $|s|$  is the energy across the space-like unit volume element per unit of time,  $\perp$  denoting the projector orthogonal to  $u$ .

Note that  $\rho$  and  $|s|$  are *relative-to-the-observers* quantities (i.e. not invariant). A simple but interesting result is that *the difference of theirs squares is an invariant quantity* [17]:

$$\rho^2 - |s|^2 = \chi^2 \quad , \quad (5)$$

where

$$\chi^2 \equiv \frac{1}{2^4}(\phi^2 + \psi^2) \quad . \quad (6)$$

We see that all the observers for which the Poynting vector vanishes see the *same* energy density  $\rho$ , that this energy density is a *minimum* and that this minimum amounts to the *invariant* quantity  $\chi$ . This is why one is naturally lead to give the following definition [18]:

**Definition:** *The invariant  $\chi$  is called the proper energy density of the electromagnetic field, and the observers that see it as their energy density, for which the Poynting vector vanishes, are said at rest with respect to the electromagnetic field.*

All other observer will see an energy density  $\rho$  corresponding to the rest energy  $\chi$  incremented by the Poynting energy  $|s|$  according to (5) [19].

The stress tensor  $\tau$  is also a relative-to-the-observer quantity, related to the Poynting vector and to the energy density by the eigen-value equation:

$$i(s)\tau = \rho s \quad . \quad (7)$$

A consequence of the above relation is that, in spite of the relative-to-the-observer character of all the elements of this equation, the other two eigen-values of  $\tau$  are invariants. They just amount, up to a sign, the proper energy density:  $\pm\chi$  [18].

c) Among all the electromagnetic fields  $F$ , there exists a particularly important class, the *pure radiation electromagnetic fields*. Usually they are physically characterized as those for which no observer sees a vanishing Poynting vector, or alternatively as those such that the electric and magnetic components are orthogonal and equimodular. But, by its physical clarity I prefer the following one [20].

**Definition:** *An electromagnetic field  $F$  is a pure radiation field if its proper energy density vanishes,  $\chi = 0$ , or alternatively if the whole energy density is radiated as Poynting energy,  $\rho = |s|$ .*

All these definitions are equivalent and still equivalent to any of the following relations:

$$s = \rho n \ , \ \tau = \rho n \otimes n \ , \ F = \ell \wedge p \ , \quad (8)$$

where  $n$  denotes the unit vector in the direction of the Poynting vector and  $\ell$  and  $p$ ,  $i(\ell)p = 0$ , define respectively the *principal direction* and the *polarization* of  $F$ .

Two-forms of the form  $F = \ell \wedge p$  are called *null*. For this reason, pure radiation electromagnetic fields are also called *null fields*.

It is important to note that for every observer  $u$  there always exist a vector  $\ell_u$  in the principal direction of the null field such that  $F = \frac{1}{\rho}(\ell_u \wedge e)$ . It follows  $\ell_u = \rho u + s$ , which gives the physical interpretation of the principal direction: it is the null lift of the Poynting vector  $s$  with respect to the observer  $u$ .

d) The non null electromagnetic fields are called *regular*, and represent (radiating or not) electromagnetic fields with Coulombian part. They at rest or proper energy density  $\chi$  never vanishes,  $\chi \neq 0$ , and they are of the form

$$F = \alpha \ell \wedge m + \beta * (\ell \wedge m) \ , \quad (9)$$

where  $\ell$  and  $m$ , are normalized null vectors, i.e.:

$$\ell.\ell = m.m = 0 \ , \ \ell.m = 1 \ , \quad (10)$$

defining the *principal directions* of  $F$ , and  $\alpha$  and  $\beta$  are simply related to  $\phi$  and  $\psi$  :

$$\phi = 2(\alpha^2 - \beta^2) \ , \ \psi = -4\alpha\beta \ . \quad (11)$$

In terms of them, the electromagnetic proper energy density  $\chi$  is given by

$$\chi = \frac{1}{2}(\alpha^2 + \beta^2) \ . \quad (12)$$

We have seen that the principal direction  $\ell$  of a null field is given by the null lift of the Poynting vector.

What is now, in the regular case, the physical interpretation of the principal directions defined by  $\ell$  and  $m$ ? What is their relation with the Poynting vector?



Paradoxically, these questions seem to have been never asked. In order to answer them, but also to control on  $F$  specific properties of  $\ell$  and  $m$ , it is worthwhile to solve *intrinsically, covariantly and explicitly*, the eigenvector problem related to these principal directions [21].

The answer is generated by the operator  $C$  [22], given in the following proposition.

**Proposition 1** (Coll-Ferrando) : *The principal directions  $\{\ell\}$ , and  $\{m\}$  of a two-form  $F$  are given by the vectors*

$$\ell = C(x) \ , \ m = {}^t C(x) \ , \quad (13)$$

where the operator  $C$ , called the principal concomitant of  $F$ , is given by

$$C \equiv \alpha F - \beta *F + T + \chi g \ , \quad (14)$$

${}^t C$  is its transposed and  $x$  is an arbitrary time-like direction.

Now, taking  $x = u$  in (13) one can prove the following results [20]:

**Proposition 2** (Coll-Soler): *The invariant directions  $\{\ell\}$  and  $\{m\}$  of a regular electromagnetic field  $F$  are given by the null shifts  $\ell_u$  and  $m_u$  respectively of the vectors*

$$\ell_\perp = s - r \ , \ m_\perp = s + r \ , \quad (15)$$

where  $s$  is the Poynting vector and  $r$  is given by:

$$r \equiv \alpha e + \beta h \ . \quad (16)$$

**Corollary:** *The Poynting vector  $s$  with respect to an observer  $u$  of an electromagnetic field  $F$  is half the sum of the projection of the invariant principal vectors  $\ell_u$  and  $m_u$  of  $F$ :*

$$s = \frac{1}{2}[\ell_\perp + m_\perp] \ . \quad (17)$$

This corollary strongly suggests a privileged decomposition of an electromagnetic field into two pure radiation components. In fact, one can prove the following result:

**Proposition 3** (Coll-Soler): *For every observer  $u$ , the energy tensor  $T$  of any regular electromagnetic field  $F$  may be obtained univocally as the composition of two pure radiation energy tensors along the principal directions of the field,*

$$T = \frac{1}{2\chi}(T_{\ell_u} T_{m_u} + T_{m_u} T_{\ell_u}) - \chi g \quad (18)$$

where  $T_{\ell_u}$  and  $T_{m_u}$  are given by

$$T_{\ell_u} = \frac{2}{(\rho + \chi)} \ell_u \otimes \ell_u \ , \ T_{m_u} = \frac{2}{(\rho + \chi)} m_u \otimes m_u \ . \quad (19)$$

This result allows to say that the principal directions of a regular electromagnetic field are the null shifts of the Poynting vectors corresponding to the two pure radiation electromagnetic fields whose composition (18) generates the field.

e) From a technical point of view, Proposition 1 constitutes a very interesting tool. It allows to solve inverse problems concerning the necessary and sufficient conditions to be verified by a tensor in order to insure particular differential properties of some of its eigenspaces. Up to now, the only known result on such problems corresponded to a very simple situation [23]. As an example, we give here for the electromagnetic field the following one, related to the permanence of a pure radiation field [22]:

**Proposition 4** (Coll-Ferrando): *An electromagnetic field  $F$  has a geodesic principal direction if, and only if, its principal concomitant  $C$ , given by (14), satisfies*

$$\text{tr}\{C \wedge i(C)\nabla C\} = 0 \quad . \quad (20)$$

where  $C$  is considered as a vector valued one-form.

In the last Section, similar techniques have been used to find the non-linear generalization of Maxwell equations.

f) The real vector space structure of the solutions of Maxwell equations means physically:

- that Maxwell equations admit a *charge-scaling law* for every electromagnetic field, and that this law is multiplicative, and
- that Maxwell equations admit a *superposition law* for every two electromagnetic fields, and that this law is additive.

The construction of a new electromagnetic field theory involves, sooner or later, to ask about the existence of these two laws, as well as about their respective multiplicative and additive character.

For “not-everything” theories, as would be the case of those we are talking about here, the existence of such laws may be epistemically required. It is thus their non-linear character that has to be specified. Let us write them respectively in the form

$$\mu \bullet F = f(\mu, F) \quad , \quad F \oplus G = g(F, G) \quad . \quad (21)$$

Whatever be the tensor character of the field, the above equations may be submitted to restrictions coming from desired or suspected conditions. For example, the parameter  $\mu$  in (21) may be restricted to take discrete values corresponding to multiples of a elementary charge; the function  $f$  be such that  $f(-1, F) = -F$  for any  $F$ ; or the function  $g$  be symmetric in its arguments etc. But here we are interested only in the general implications imposed by the two-form character of the electromagnetic field  $F$ . The corresponding general expressions are given by the following proposition [24].

**Proposition 5** (Coll-Ferrando): *i) The more general charge-scaling law for an electromagnetic two-form  $F$  is of the form*

$$\mu \bullet F = m_F F + m_{*F} *F \quad (22)$$

where  $m_F$  and  $m_{*F}$  are functions of the parameter  $\mu$  and of the two invariant scalars of  $F$ . ii) The more general superposition law for two electromagnetic two-forms  $F$  and  $G$  is of the form

$$\begin{aligned} F \oplus G &= p_F F + q_G G + r_{[F,G]} [F, G] \\ &+ p_{*F} *F + q_{*G} *G + r_{*[F,G]} *[F, G] \end{aligned} \quad (23)$$

where  $p_F, \dots, r_{[F,G]}$  are functions of the two invariant scalars of  $F$ , of the corresponding scalars of  $G$ , and of the two mixed scalars  $\text{tr} FG$  and  $\text{tr} F *G$ , and  $[F, G]$  is the commutator of  $F$  and  $G$ ,  $[F, G] \equiv FG - GF$ .

The proof of expression (22) follows from the fact that  $\{F, *F\}$  is a basis, in the module of the two-forms over the functions, for the powers of  $F$ , and that of expression (23) follows from the fact that the six two-forms of its second member form a basis for the Lie algebra generated by  $F$  and  $G$  [25].

## 5 Electromagnetic Field as a Lorentz Tensor

a) Some generic situations seem to indicate that the linearity of the electromagnetic field equations is due to the weakness of these fields in our ordinary experimental conditions.

Usually, those that take seriously this idea, try to apply more or less reasonable criteria to find non linear equations for the electromagnetic two-form  $F$ . This is to forget that, if a weak electromagnetic field is represented by the two-form  $F$ , strong electromagnetic fields may change not only its intensity, but even its tensor character.

Formally speaking, the first examples of such situations are given by Group Theory: (the linear space of) the algebra of a group is nothing but the set of “weak elements” of the group.

In this sense, the electromagnetic case is very suggestive: Maxwell equations, which are at the basis of Special Relativity, structure the electromagnetic fields at every point as anti-symmetric tensors, just like the elements of the algebra of the Lorentz group that generates Special Relativity.

Thus, at least from a formal point of view, one is lead to test the idea that the “good” strong electromagnetic fields ought to be represented by Lorentz tensor fields  $L$ , which, in the case of little intensity, would reduce to simple two-forms  $F$ .

The following two paragraphs do not pretend to present the first ingredients of a new electromagnetic theory (which nevertheless is in progress) but only to illustrate by a formal example the general idea that *strong fields may change the tensor character of a weak field representation*, and to show that this idea is workable and interesting in some of its consequences.

b) Thus, let us accept that strong electromagnetic fields are correctly represented by Lorentz tensor fields [26]. Then, “near” the Maxwellian weak electromagnetic solutions (technically: in the exponential domain of the Lorentz Group), Lorentz tensors  $L$  repre-

senting generic electromagnetic fields are related exponentially to their weak counterpart  $F$  :  $L = \exp F$  .

The exponential and the logarithmic branches allow to exchange the variables  $F$  and  $L$  in the exponential domain, and also the corresponding equations. Nevertheless, the solutions to these differential equations in terms of Lorentz tensors will belong, not only to the exponential domain, or to the rest of the connected-to-the-identity component but even to whole disconnected group. The Lorentz tensors corresponding to these last regions are not the exponential of two-forms, so that, even imposing to  $L$  the transformed Maxwell equations, we will have much more electromagnetic solutions than the exponential of the Maxwell ones.

But the new solutions corresponding to the disconnected components will never reduce continuously, when varying the integration constants, to the identity, i.e. will never become vanishing electromagnetic fields. The possibility of the physical existence of such fields is by itself very interesting (see paragraph **2.b**).

Let us already note that, for not strong electromagnetic fields  $L$  , for which two-forms  $F$  exist such that  $L = \exp F$  , one has  $L = g$  for vanishing  $F$  . But the metric  $g$  is nothing but the *inertia or gravitational tensor field* of Minkowski or of general Riemann space-times respectively: *Lorentz tensor fields are objects that may simultaneously describe electromagnetic and gravitational fields.*

For the above reasons, I believe that such fields are worthy of deeper analysis.

**c)** The basic elements for such an analysis are the functions  $\exp$  and  $\ln$  . To handle them in the context of a field theory, it is imperative to sum their usual infinite series expansion.

Paradoxically, the sum of these series is still an open problem even for many simple groups. For the sum of the exponential series for the Lorentz group one has [27]:

**Proposition 6** (Coll-Sanjosé): *The exponential of a two-form  $F$  in a space-time of metric tensor  $g$  is given by*

$$\exp F = \epsilon_g g + \epsilon_T T + \epsilon_F F + \epsilon_{*F} *F , \quad (24)$$

where

$$\begin{aligned} \epsilon_g &= \frac{1}{2}(\cosh \alpha + \cos \beta) \\ \epsilon_T &= \frac{1}{\alpha^2 + \beta^2}(\cosh \alpha - \cos \beta) \\ \epsilon_F &= \frac{1}{\alpha^2 + \beta^2}(\alpha \sinh \alpha + \beta \sin \beta) \\ \epsilon_{*F} &= \frac{1}{\alpha^2 + \beta^2}(-\beta \sinh \alpha + \alpha \sin \beta) \end{aligned} \quad (25)$$

For the sum of the logarithmic series we present here, for simplicity, only the result for non symmetric proper Lorentz tensors on the principal branch [28].

**Proposition 7** (Coll-Sanjosé): *The logarithm of a non symmetric proper Lorentz tensor field  $L$  is given by*

$$\ln L = h_{aL} {}^a L + \epsilon h_{*aL} {}^a L \quad (26)$$

where  ${}^aL$  is the anti-symmetric part of  $L$ ,  $\epsilon$  is the sign of the scalar  $\text{tr} {}^aL {}^aL$ , and  $h_{aL}$  and  $h_{*aL}$  are the functions

$$\begin{aligned} h_{aL} &= \frac{1}{\mu^2 - \nu^2} \{(\mu^2 - 1)^{1/2} \arg \cosh \mu + (1 - \nu^2)^{1/2} \arccos \nu\} \\ h_{*aL} &= \frac{1}{\mu^2 - \nu^2} \{(1 - \nu^2)^{1/2} \arg \cosh \mu - (\mu^2 - 1)^{1/2} \arccos \nu\} \end{aligned} \quad (27)$$

of the invariant scalars  $\mu$  and  $\nu$  of  $L$  given by

$$\begin{aligned} \mu &\equiv \frac{1}{4} \{ \text{tr} L + \sqrt{2 \text{tr} L^2 - \text{tr}^2 L + 8} \} \\ \nu &\equiv \frac{1}{4} \{ \text{tr} L - \sqrt{2 \text{tr} L^2 - \text{tr}^2 L + 8} \} . \end{aligned} \quad (28)$$

In this scheme, some simple charge-scaling and superposition laws on proper tensors homologous to the linear ones on their associated two-forms may be imposed. The more natural ones are, of course, those of the Lorentz group structure:

$$\lambda \bullet L = \exp\{\lambda \ln L\} , \quad L \oplus M = L \times M , \quad (29)$$

where  $\times$  denotes the cross product [29]. The expression of this last law in terms of the associated two-forms is known as the BCH-formula (Baker-Campbell-Hausdorff), and its explicit and covariant summation for the Lorentz group has been recently given. This summation is, as it is due, of the form (23) were the particular values of the scalar coefficients  $p_F, \dots, r_{*[F,G]}$ , may be found in [30].

## 6 Quadratic Electromagnetic Field Equations

**a)** This Section presents a non-linear pure field theory of electromagnetism, very likely the nearest to Maxwell theory. As this last one, our non-linear theory supposes that the electromagnetic field is a two-form  $F$ , but our field equations for it turn out to be quadratic.

I consider this theory as slightly better than Maxwell one, because it contains exactly the additional solutions we wanted to have. But apart from this fact, it inherits all the other bad aspects of Maxwell theory, particularly the duality invariance and the advanced-retarded symmetry. In spite of that, I believe it is worthwhile to present it: as a slightly improvement on Maxwell theory, of course, but also as an illustration of some of the concepts and ingredients above mentioned and, overall, because of its striking properties.

**b)** At the exterior of sources, Maxwell equations for an electromagnetic two-form  $F$  are

$$dF = 0 , \quad \delta F = 0 , \quad (30)$$

where  $d$  and  $\delta$  denote respectively the exterior differential and the divergence (up to sign) operators.

For regular fields, expression (9) may also be written

$$F = \alpha U + \beta *U \quad (31)$$

where  $U$  is a unit two-form,

$$\text{tr } U^2 = 2 \quad , \quad \text{tr } U *U = 0 \quad , \quad (32)$$

representing the *induced metric volume* on the time-like two-plane  $U$  of vectors  $x$  such that  $i(x) *U = 0$ . Maxwell equations admit for the weights  $\alpha$  and  $\beta$  the *conditional system* in  $U$  [31]

$$\begin{aligned} \delta [\delta U \wedge U - \delta *U \wedge *U] &= 0 \quad , \\ \delta [\delta U \wedge *U + \delta *U \wedge U] &= 0 \quad , \end{aligned} \quad (33)$$

from which  $\alpha$  and  $\beta$  are determined up to a constant related to initial values. Time-like two-planes  $U$  verifying (33) are called *Maxwellian*, because they generate *all* the regular solutions to Maxwell equations. Equivalently, Maxwell equations for regular electromagnetic fields may be written, modulo initial conditions, in the Rainich energy form [32]

$$\left. \begin{aligned} \text{tr } T &= 0 \quad , & T^2 \wedge g &= 0 \\ \delta T &= 0 \quad , & d \frac{*(T \times \nabla T)}{\text{tr } T^2} &= 0 \end{aligned} \right\} \quad (34)$$

**c)** It is important to note that neither equations (33), nor equations (34) are valid when  $F$  is null. For a null or pure radiation field,  $F = \ell \wedge p$ , the eight Maxwell equations (30) group in *four* sets of *two* equations implying and only implying the following properties:

- the null direction is geodesic,
- the polarization is parallel transported along the null direction,
- the null direction is distortion-free,
- the gradient of the energy density is specifically related to the polarization vector.

**d)** This set of coupled equations appear as excessively restrictive in some frequent situations.

It is the case in theoretical studies on wave-guides, where some authors [33] claim that Maxwell equations have an insufficient number of pure radiation solutions.

Also, in Special Relativity, meanwhile spherically symmetric pure radiation electromagnetic fields are forbidden for evident topological reasons, Maxwell equations are the sole responsible of the banning of cylindrically symmetric pure radiation fields, among others.

Even worse, there exists no vacuum gravitational space-times in which Maxwell equations admit generic pure radiation solutions; in other words, a torch cannot bring light in General

Relativity. This is due to the Bel-Goldberg-Sachs theorem [34], that reduces drastically the existence of shear-free geodesic null directions in curved space-times.

e) Pure electromagnetic fields, in particular plane waves, like inertial observers, free particles or free fields, are paradigmatic concepts in physics. Their existence can only be proved locally and with rough precision, but their importance resides both, in the simple concepts that they involve and in the non-trivial constructions that they are able to provide.

On the other hand, the extraordinary success of Maxwell equations in Physics is so enormous, that any answer allowing them to save their form or their structure is in general preferred. Thus, there exist some theoretical answers to the above anomalies, still making Maxwell equations unchanged. But we shall suppose here what is *the more direct* conclusion from the above three points, namely that:

*Maxwell equations contain insufficient pure radiation electromagnetic fields.*

Of the above four pairs of Maxwell equations for the null case, the third pair is the responsible for the non existence of cylindrical waves in Minkowski and generic waves in curved space-times, and the fourth restricts severely the number of solutions to those imposing a particular relation between the orientation of the polarization and the intensity of the field. There are these restrictions those that prevent to identify electromagnetic pure radiation fields with beams of electromagnetic rays. This is why we propose to substitute Maxwell equations  $\mathbf{M}(F)$  by a new set of equations  $\mathbf{S}(F)$  restricted to the following *schedule of conditions*:

- i. the pure radiation electromagnetic field solutions of the new equations  $\mathbf{S}(F)$  must be *all* those having:
  - their principal direction geodetic,
  - their polarization parallel propagated,
- ii. the regular electromagnetic field solutions to the new equations  $\mathbf{S}(F)$  must differ as little as possible from the corresponding regular solutions to Maxwell equations  $\mathbf{M}(F)$ .

f) There exists a natural class of operators on algebras, called derivations. A derivation  $\mathbf{d}$  of an algebra  $(+, \diamond)$  is an operator that verifies the Leibniz rule for the product,  $\mathbf{d}(f \diamond g) = \mathbf{d}f \diamond g + \epsilon_f f \diamond \mathbf{d}g$ , where  $\epsilon_f$  is the parity sign of the element  $f$ . Leibniz rule allows to associate to any other operator  $\mathbf{d}$  on the algebra an internal binary composition law  $\{f, g\}$ , that we call the *Leibniz bracket* of  $\mathbf{d}$  with respect to the algebra  $(+, \diamond)$ , by means of the relation  $\{f, g\} + \mathbf{d}(f \diamond g) = \mathbf{d}f \diamond g + \epsilon_f f \diamond \mathbf{d}g$ . So, one can say that, for a given algebra, an operator is a derivation iff its Leibniz bracket vanishes.

It is well known that the exterior derivative  $d$  is a derivation of the exterior algebra  $(+, \wedge)$ , but that the divergence operator  $\delta$  is not.

**Definition:** The Schouten bracket  $\{F, G\}$  of two exterior forms  $F$  and  $G$  is the Leibniz bracket of the divergence operator  $\delta$  with respect to the exterior algebra:

$$\{F, G\} \equiv \delta F \wedge G + (-1)^p F \wedge \delta G - \delta(F \wedge G) \quad (35)$$

where  $p$  is the parity of  $F$ .

With this instrument, and applying concepts of the preceding Sections to write down these specifications in terms of the electromagnetic field itself, one can proof:

**Proposition 8** (Coll-Ferrando): *The equations  $\mathbf{S}(F)$  on the electromagnetic field two-form  $F$  that satisfy the above schedule of conditions are:*

$$\mathbf{S}(F) \equiv \begin{cases} \delta [F^2 + (*F)^2] = 0 \\ \{F, F\} + \{*F, *F\} = 0 \end{cases} , \quad (36)$$

where  $\{, \}$  is the Schouten bracket.

**g)** Our new electromagnetic field equations  $\mathbf{S}(F)$  verify the schedule of conditions in the *strongest* sense: all the null two-forms with geodetic principal direction and parallel propagated polarization are solutions of them, and their regular solutions are *exactly* the regular Maxwellian ones, in spite of the apparent difference between our equations (36) and the Maxwell ones (30). Denoting by  $\Sigma(\mathbf{S})$  and  $\Sigma(\mathbf{M})$  respectively the space of solutions of our system and that of Maxwell equations, and by  $\{F_N\}$  the set of null two-forms with geodetic principal direction and parallel propagated polarization, one has

$$\Sigma(\mathbf{S}) = \Sigma(\mathbf{M}) \cup \{F_N\} \quad (37)$$

Thus our new equations strictly make nothing but to add to Maxwell equations the up to now missing pure radiation solutions. But it is important to note that, if the charge-scaling law remains the usual product by a number, now the superposition law with ingredients in  $\{F_N\}$  is no longer additive. In spite of our result (23) of Proposition 5, for the moment we have been unable to found it.

Maxwell succeed in formulating the action at a distance laws of Coulomb, Biot-Savart, Ampere and Faraday in terms of the electromagnetic force fields (adding its displacement current), obtaining his celebrate equations [35]. Imagine for a moment he tried to formulate them, in terms of the energetic fields (energy density, Poynting vector, stress tensor), instead of in terms of the force fields. Then, he would obtained, instead of his equations, the Rainich ones (34), valid only for regular fields; in other words, *he would not discovered the electromagnetic character of the light!*

The last of Rainich equations (34) is indeterminate for null energy tensors, but, with the help of null two-forms, and expressed in term of them, this indeterminacy may be *solved*, the result being *our equations* (36). In other words, *Maxwell could discovered our equations!*

Our equations are the necessary and sufficient conditions for null principal directions to be geodetic with a parallel transported polarization. These are the basic ingredients of the



geometric optics approximation. But Maxwell equations are nothing but the *same* equations for *regular* fields, so that we have:

*the exact equations of the geometric optics approximation of Maxwell equations for null fields, are the exact Maxwell equations for regular fields.*

This result may be considered as a classical version of Feynman's point of view on quantum electrodynamics.

## 7 Conclusion

A little number of the many unclear aspects enveloping electromagnetism have been commented. Some of them concern, more generally, the very notion of (pure) field theory and the form and properties of its equations (Section 2). And others affect the double aspect, theoretical and experimental, of the tensor character and texture of the electromagnetic field itself (Sections 3 and 4). Its invariant elements, proper energy density, observers at rest, space-like principal directions, which paradoxically remained until now mere mathematical variables, have been physically analysed (Section 4).

It has been shown that strong fields could not only increase the intensity of the ordinary ones but also change their tensor character; and the example of Lorentz tensors, which, although formal, shows the main features of this eventuality, has been presented (Section 5).

Finally, a physical application of some of the techniques presented for the study of the texture of the electromagnetic field has been given: an electromagnetic theory that generalizes that of Maxwell, providing “more light” than that contained in it, but rigorously respecting all the not purely radiation ones (Section 6). The new solutions could make our theory (more) complete, if they were detected by appropriate experiments; otherwise, lacking in physical meaning, they would make it less strict. But in any case, the new theory suffers, at least, of the same level of non-strictness than Maxwell theory.

The problem of finding a complete and strict field theory of electromagnetism remains therefore open. It has not been possible to develop here other equally important unclear aspects of the current field theories, but they constitute so many reasons of dissatisfaction raised by the current notion of “field theory”.

The history of science reveals not only the brilliant evolution of some ideas, but also and abundantly, the bad or null evolutions of many others. It is the task of the physicist, theorist or experimental, to correct this situation. I hope that the simple results presented here be an incentive in this direction.

## References

- [1] Many of the results and ideas included here are the reflection of a friendly and fruitful collaboration for a very long time with L. Bel, J.J. Ferrando, J.A. Morales, F. San José

and A. Tarantola.

- [2] We are here interested by the electromagnetic fields in vacuum, irrespective of the sources that produce them. The sources will be located at, or around, the singularities of these vacuum fields, and an important but unsolved problem of *all* the usual field theories is to find an algorithm allowing to locate, from the knowledge of the *regular, exterior field on a local, finite, region*, its singularities and the values of the charges associated to them. To our knowledge, at present such an algorithm is only known for the very simple gravitational theory of *one* Newtonian point particle (see reference [10]).
- [3] The other extreme point of view considers straightaway electromagnetism in presence of matter, involving fields, inductions, charge densities, currents and constitutive equations. One of the best descriptions of this scenario, based in a particularly elegant axiomatics, may be found in F. W. Hehl and Y. N. Obukhov, *Foundations of Classical Electrodynamics*, in press.
- [4] We work here with this *pure field* notion better than with a matter model because in this last case the equations obtained, on one hand, contain the vacuum ones and, on the other, are “closed” by the constitutive equations. This amounts to say that they are equivalent to the vacuum ones plus a particular class of electromagnetic matter models, so that the “texture part” of them is equally concerned by our purposes.
- [5] B. Coll, *A Universal Law of Gravitational Deformation for General Relativity*, in Proc. of the ERE-98 Spanish Relativity Meeting in honour of the 65th Birthday of Lluís Bel “Gravitation and Relativity in General” ed. J. Martín et al., World Scientific (1999). See also <http://coll.cc>.
- [6] The possibility to detach from particles non-scaping parts of the field, may be related to the existence, for Lorentz tensors, of four disconnected components. Only the one containing the unit Lorentz tensor is directly related to two-forms, that is to say, to ordinary electromagnetic fields.
- [7] B. Coll and J.J. Ferrando, *Non Linear Maxwell Equations* in Proc. of the ERE-93 Spanish Relativity Meeting “Relativity in General” Éditions Frontières, (1995).
- [8] “Very null” is used here not as a mathematical concept, but as an intuitive one, that wishes to express, as the arguments that follow the text show, that solutions describing physical phenomena constitute a set much lesser than the “natural” subsets of null measure in the corresponding spaces.
- [9] That means that, in the space of solutions, for every point belonging to  $\mathbf{F}$ , i.e. representing a physical field, there are  $2^N - 1$  points in the gravitational case and  $R^N$  in the electromagnetic one, that belong to  $\hat{\mathbf{S}}$ , i.e. that are unphysical solutions.
- [10] B. Coll and J.J. Ferrando, *The Newtonian Point Particle*, in Proc. of the ERE-97 Spanish Relativity Meeting “Analytical and Numerical Approaches to Relativity Sources

of Gravitational Radiation” ed. by C. Bona et al., Univ. de les Illes Balears, Spain (1998). See also <http://coll.cc>.

- [11] Already in the first two corollaries of his Principia, Newton argues for the law of the parallelogram for the composition of forces, and asserts that this law is abundantly confirmed by mechanics. I do not know any experiment conceived specifically to measure this concordance of parallelograms and forces.
- [12] Induced by an epistemic analysis of vacuum Maxwell equations and the equivalence principle for inertial observers. See the following paragraph c).
- [13] They are the specific formalisms for linear spaces that oblige to a drastic change when passing from the cosine law to any other one. But it is easy to see that in a general space of functions of a given direction, the cosine law is dense in analytically related ones.
- [14] In local charts  $e_\alpha = u^\rho F_{\rho\alpha}$ ,  $h_\alpha = \frac{1}{2}u^\rho \eta_{\rho\alpha\sigma\tau} F^{\sigma\tau}$ ,  $\eta_{\alpha\beta\gamma\delta}$  being the volume element of the metric. Reciprocally,  $F_{\alpha\beta} = u_\alpha e_\beta - u_\beta e_\alpha - \eta_{\alpha\beta\rho\sigma} u^\rho h^\sigma$ .
- [15] It is to be noted that, although we call  $F$  the *electromagnetic field*, this is *in fact* a short cut for *electro-magnetic-observer field*: as the expression  $F(e, h, u)$  of the text reveals, one cannot extract from  $F$  the physical electromagnetic information unless we fix the observer  $u$ . In fact, one can show that, in the generic case, the equation  $F = F(e, h, u)$  for the three variables  $u, e, h$ , admits, for every arbitrary choice of one of them, a unique solution in the other two.
- [16] That any tensor over a vector space on a field  $K$  is nothing but a graph of collections of vectors *weighted* by elements of  $K$ , is logically trivial and algorithmically unexplored, up for second order tensors of any type and third order tensors of the class of the structure constants of a group. But both, for mathematical as well as physical applications, it would be very interesting to develop algorithms and canonical forms (extensions of the Jordan form) for tensors of any order and type.
- [17] We use everywhere  $c=1$ . In arbitrary units  $[\rho] = [|s|][c]^{-1}$ .
- [18] B. Coll and D. Soler, *Algebraic properties of the electromagnetic field*, unpublished work.
- [19] Observe that the proper energy density and the Poynting energy contribute to the total energy in a form that, although quadratic, is similar to that of the proper mass and kinetic energy in the mechanical case. The difference is that, meanwhile for mechanical systems there exists only *one* observer at rest, for an electromagnetic field there exists a *one-parameter* family.
- [20] B. Coll and D. Soler, *The radiation parts of an electromagnetic field*, unpublished work.

- [21] Intrinsic (i.e. expressed in terms of the sole quantities present in the statement), covariant (i.e. coordinate free) and explicit (i.e. allowing to be checked by direct substitution of the variables in the proposed expressions) solutions to geometric or physical problems, or *ICE solutions*, are very infrequent, in spite of their great conceptual interest and practical advantages. The present situation is an example. Surprisingly enough, in spite of the old origin of linear analysis and the intense use of matrix and tensor calculus in many branches of science, the general solution to the eigenvector problem for matrices has been considered but recently (note that we speak about the *general solution*, i.e. the general expression, depending on the matrix, of the invariant spaces corresponding to every eigenvalue); see C. Bona, B. Coll and J. A. Morales, J. Math. Phys., **33** (2) p. 670 (1992), for the four-dimensional symmetric case, [22] for the corresponding anti-symmetric case, and G. Sobczyk, The College Mathematical Journal, **28** p. 27 (1997), for the general n-dimensional case.
- [22] If an electromagnetic field is a pure radiation field at an instant (say, a plane wave), do Maxwell equations insure that it will remain a pure electromagnetic field in subsequent times? In contrast with a common opinion, the answer is negative. See B. Coll and J. J. Ferrando, Gen. Rel. and Grav., **20** (1) p. 51 (1988). In this paper, the general eigenvector problem for two-forms in four-dimensional space-times is solved.
- [23] J. Haantjes, Proc. Konink. Nederl. Akad. Van Wetens. Amsterdam, **A 58**(2), p. 158 (1955). He solved the problem of obtaining the necessary and sufficient conditions, for symmetric tensors of type I, insuring that *all* the eigen-vectors are integrable. What about the cases in which *not all* them are integrable, or the cases in which the tensor is not of type I, or those where the differential system considered is not the integrability one? All these questions remained open until now.
- [24] B. Coll and J. J. Ferrando, *Superposition Laws for the Electromagnetic Field*, unpublished work.
- [25] B. Coll and F. San José, J. Math. Phys., *37*, p. 5792 (1996); see also *Relative positions of a pair of planes and algebras generated by two 2-forms in relativity*, in *Recent Developments in Gravitation*, World Scientific (1991).
- [26] In the present context it is supposed that the electromagnetic fields are able to be described by adimensional quantities. The ways and physical meanings that allow this situation are theory-dependent, and will not be discussed here.
- [27] B. Coll and F. San José, Gen. Relativity and Grav., **22** p. 811 (1990).
- [28] For any proper Lorentz tensor and any branch, see [27]
- [29] Contraction over the adjacent spaces of their tensor product, i.e. the matrix product of their corresponding components.
- [30] B. Coll and F. San José, Gen. Relativity and Grav., **34** (9), p. 1345 (2002).

- [31] B. Coll, F. Fayos and J. J. Ferrando, J. Math. Phys., **28** (5), p. 1075 (1987).
- [32] G. Y. Rainich, Trans. Am. Math. Soc., **27** p.106 (1925).
- [33] For a review, see H.F. Harmuth, *Propagation of Non-sinusoidal Electromagnetic Waves* and also *Radiation of Non-sinusoidal Electromagnetic Waves*, Academic Press, 1986 and 1990 respectively)
- [34] L. Bel, Cahiers de Physique, *16* p.59-80 (1962); J.N. Goldberg and R.K. Sachs, Acta Phys. Polon., Suppl. *22* p. 13 (1962).
- [35] Observe that, if he tried to be rigorous in this correspondence, he would be obliged to joint to its linear equations the empirical quadratic inequalities  $(e^2 - h^2)^2 + (e.h)^2 \neq 0$ , verified in all the situations implied by the above mentioned four laws. This empirical rigour not only would break the linearity of his equations but would forbidden the existence of electromagnetic waves!